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Existence of States on Pseudoeffect Algebras

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Pseudoeffect (PE) algebras have been introduced as a noncommutative generalization of effect algebras. We study in this paper PE algebras with the special property of having a nonempty state space. To this end, we consider PE algebras which are *po*-group intervals and which are, in a certain sense, noncommutative only in the small. Such a PE algebra is shown to possess a nontrivial commutative homomorphic image from which then follows that there exist states. A typical example is given by an interval of the lexicographical product of two *po*-groups the first of which is abelian.

KEY WORDS: pseudoeffect algebras; *po*-groups; PE algebras with Riesz properties; states on PE algebras.

1. INTRODUCTION

MV algebras, introduced by Chang in the 1950s, provide an algebraic semantics for the Łukasiewicz multivalued logics (Cignoli *et al.*, 2000). Not long ago, Georgulescu and Iorgulescu (2001) introduced a noncommutative counterpart of MV algebras, the pseudo-MV algebras, heading towards the conception of a logic which in general does not allow to interchange the two arguments of a conjunction. MV- as well as pseudo-MV algebras are intervals of ℓ-groups, in the first case of the abelian ones (Dvurečenskij, 2002a; Mundici, 1986).

Effect algebras, introduced in 1993 (Foulis and Bennett, 1994), generalize MV algebras. They are partial additive algebras modeled upon the Hilbert space quantum effects, which in turn represent the positive outcomes of the yes–no tests performable at some physical system (Busch *et al.*, 1995). Any interval of an abelian *po*-group, not necessarily a lattice-ordered one, gives rise to an effect algebra.

Now, pseudoeffect algebras—or PE algebras for short—have been recently introduced in Dvurečenskij and Vetterlein (2001a). They generalize effect algebras in that they are no longer necessarily commutative; and the same time, they contain

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pseudo-MV algebras as a subclass. PE algebras arise from intervals of *po*-groups of any kind; if this is the case, we talk about interval PE algebras. In this paper, we are focussed exclusively on this kind of PE algebra.

The dependences among the mentioned structures is illustrated by the following scheme. Note that there is a corresponding scheme for *po*-groups: containing *po*-groups, abelian *po*-groups, ℓ-groups, and abelian ℓ-groups.

interval pseudoeffect (interval PE-) algebras $(E; +, 0, 1)$	$\Leftarrow$	interval effect algebras $(E; +, 0, 1)$ (commutative)
$\Uparrow$		$\Uparrow$
pseudo-MV algebras $(E; \oplus, ^{-}, ^{\sim}, 0, 1)$ (lattice ordered, fulfilling (RDP))	$\Leftarrow$	MV algebras $(E; \oplus, ^{*}, 0, 1)$ (lattice ordered, commutative, fulfilling (RDP))

The interval PE algebras are probably not characterizable in a purely algebraic manner. But by Dvurečenskij and Vetterlein (2001b), there is certain kind of Riesz decomposition property, called (RDP_1), which implies for a PE algebra to be an interval PE algebra.

We are working towards a structure theory for interval PE algebras. In Dvurečenskij and Vetterlein (2001c), congruences have been considered and in particular, states have been examined to a certain extent. This paper may be understood as a continuation of this discussion.

Given an interval PE algebra, the first question to be asked might be, if there is a state at all. This question has already been considered for the subclasses of PE algebras shown in our scheme. States on MV algebras were studied in Chovanec (1993) and Mundici (1995); in particular, MV algebras always possess states. The same is true for interval effect algebras (Goodearl, 1996), although it is not true in general for effect algebras (Greechie, 1971; Riečanová, 2001). In the noncommutative case, the situation is more difficult. States reflect the commutative part of an algebra only; so it is not amazing, although far from easy to see, that even a pseudo-MV algebra need not have any state (Dvurečenskij, 2001). An algebraic criterion which exactly would tell that at least one state exists, is not known.

This paper is organized as follows. After providing in Section 2 a condensed version of all the background material needed, we discuss in Section 3 in a first step the exact condition for a PE algebra to possess a state. Here, we in particular generalize the result of Goodearl (1986) about abelian interpolation groups.

In what follows, we are guided by the following idea. Rather than further considering the general case, we ask which classes of PE algebras seem interesting and reasonable to be studied at all in the present context. In a first step, in Section 4, we will actually study a class of concrete examples—intervals in the lexicographical product of abelian *po*-groups and possibly non-abelian ones. For them, we give the exact conditions to fulfil the Riesz decomposition property.

These lexicographical products may be considered the typical examples of PE algebras being noncommutative only in the small. In Section 5, this idea is developed on general algebraic grounds. We will define order-regular and nearly commutative PE algebras. We will prove that algebras of this kind always possess a nontrivial interval effect algebra as a homomorphic image. As a particular consequence, the state space is then nonempty.

Furthermore, we introduce a slightly strengthened version of order regularity, and on the basis of it, we give an improved version of the state existence theorem. Namely, by assuming this property, we may show that a PE algebra is represented by a *po*-group whose commutator subgroup is purely infinitesimal.

2. INTERVAL PE ALGEBRAS

PE algebras have been introduced in Dvurečenskij and Vetterlein (2001a) as a generalization of effect algebras. The axiom of commutativity, holding for the latter structures, is replaced by what is called here pseudocommutativity. Consequently, also the uniqueness of a complement is no longer assumed and replaced by the uniqueness of a left and of a right complement. We recall the basic definitions.

Definition 2.1. A structure $(E; +, 0, 1)$, where $+$ is a partial binary operation and 0 and 1 are constants, is called a *pseudoeffect algebra*, or *PE algebra* for short, if, for all $a, b, c \in E$, the following holds.

(PE1) $a + b$ and $(a + b) + c$ exist if and only if $b + c$ and $a + (b + c)$ exist, and in this case $(a + b) + c = a + (b + c)$ (*strong law of associativity*).
(PE2) If $a + b$ exists, there are elements $d, e \in E$ such that $a + b = d + a = b + e$ (*pseudocommutativity*).
(PE3) There is exactly one $d \in E$ and exactly one $e \in E$ such that $a + d = e + a = 1$ (*unique left and right complementation*).
(PE4) If $1 + a$ or $a + 1$ exists, then $a = 0$ (*zero-one law*).

Furthermore, for any $a, b, \in E$, we define $a \leq b$ to hold if $a + c = b$ for some $c \in E$.

Finally, a PE algebra E is called an *effect algebra* if, for all $a, b \in E$, $a + b$ is defined if and only if $b + a$ is defined, in which case $a + b = b + a$.

By $\leq$, a PE algebra is partially ordered, and this order is, by the pseudocommutativity, two-sided.

In view of (PE3), we may introduce unary operations for the two complements. Furthermore, since a PE algebra has the cancellation property (Dvurečenskij and Vetterlein, 2001a, Lemma 1.4(v)), we may introduce a left and a right difference.

Definition 2.2. Let $(E; +, 0, 1)$ be a PE algebra. Let $^{\sim}, ^{-}$ be those unary operations on E such that, for any $a \in E$,

$$a + a^{\sim} = a^{-} + a = 1.$$

We call a PE algebra *symmetrically complemented* if $^{\sim} = {}^{-}$.

Furthermore, let $/, \backslash$ be the partial binary operations on E such that, for $a, b \in E$, b/a is defined iff $b \setminus a$ is defined iff $a \leq b$, in which case we have

$$(b/a) + a = a + (b \backslash a) = b.$$

We then call $b \setminus a$ the *left* and b/a the *right difference* of b and a.

Remark 2.3. We note that for elements a, b of a symmetrically complemented PE algebra, the existence of $a + b$ and of $b + a$ is equivalent.

We next clarify the notions of homomorphisms and congruences of PE algebras (for further details, see Dvurečenskij and Vetterlein, 2001c).

Definition 2.4. Let $(E; +, 0, 1)$ and $(F; +, 0, 1)$ be PE algebras.

- (i) A mapping $\varphi : E \to F$ is called a *homomorphism* if the constants are preserved and, whenever $a + b$ is defined for $a, b \in E$, also $\varphi(a) + \varphi(b)$ is defined and equals $\varphi(a + b)$.
- (ii) A homomorphism $\varphi : E \to F$ is called an *epimorphism* if its image $\varphi(E)$ generates the whole algebra F.
- (iii) An equivalence relation $\sim$ on E is called a *congruence* if the following holds: For any $a, a', b, b' \in E$ such that $a \sim a'$, $b \sim b'$ and $a + b$, $a' + b'$ are both defined, we have $a + b \sim a' + b'$.

A congruence $\sim$ on E is called a *PE algebra congruence* if the quotient algebra $[E]_{\sim} = \{[a]_{\sim} : a \in E\}$ is again a PE algebra. Here, the addition on $[E]_{\sim}$ is defined according to rule $[a]_{\sim} + [b]_{\sim}$ is defined iff for some $a' \sim a$ and $b' \sim b$, and equals $[c]_{\sim}$ iff $a' + b'$ is defined and $a' + b' \sim c$ for some $a, b, c \in E$, and the constants are $[0]_{\sim}$ and $[1]_{\sim}$.
A congruence does in general not lead again to a PE algebra; but we have the following (Dvurečenskij and Vetterlein, 2001c, Proposition 3.3 (ii)).

Proposition 2.5. *Let* $(E; +, 0, 1)$ *be a PE algebra, and let* $\sim$ *be a congruence on* E *such that, for any pair* $a, b \in E$ *the sum of which is defined, the following condition holds: For any* $a' \sim a$*, there is a* $b' \sim b$*, such that* $a' + b'$ *is defined; and for any* $b'' \sim b$ *there is an* $a'' \sim a$*, such that* $a'' + b''$ *is defined. Then* $\sim$ *is a PE algebra congruence.*

Note that a congruence on a PE algebra may identify any pair of elements; this results in an algebra with only one element. We shall call a PE algebra $(E; +, 0, 1)$ *nontrivial* if this is not the case, that is, if $0 \neq 1$.

Now, as explained in the Introduction, we are interested in those PE algebras which arise from intervals of *po*-groups.

Definition 2.6. A structure $(G; +, \leq, u)$, where $(G; +, \leq)$ is a *po*-group and u is strong unit of G, is called *a unital po-group*. We will usually refer to it simply by (G, u).

For a unital *po*-group (G, u), let $\Gamma(G, u) \stackrel{\text{def}}{=} \{g \in G : 0 \leq g \leq u\}$, and define $+$ to be the partial binary operation on $\Gamma(G, u)$ that is the restriction of the group addition to those pairs of elements whose sum lies in $\Gamma(G, u)$. Then the structure $(\Gamma(G, u); +, 0, u)$ is called an *interval PE algebra*.

It is easily checked that an interval PE algebra is actually a PE algebra.

Among the unital *po*-groups representing a given interval PE algebra, there is a canonical one, defined and constructed as follows (Dvurečenskij and Vetterlein, 2001c, Definition 5.2, Proposition 5.3).

Definition 2.7. Let $(E; +, 0, 1)$ be a PE algebra and let (G, u) be a unital *po*-group. Let $\iota : E \to \Gamma(G, u)$ be an injective epimorphism such that, for every mapping ψ from E into a group H preserving existing sums, there is a group homomorphism $h_\psi : G \to H$ such that $\psi = h_\psi \circ \iota$. Then (G, u) together with ι is called a *universal ambient group for E*.

Certainly, if a PE algebra has a universal ambient group, then it is the only one up to isomorphism.

Proposition 2.8. *Let* $(E; +, 0, 1)$ *be an interval PE algebra. Let* $\mathcal{U}(E)$ *be the group given by the generators* E *and the defining relations* $a + b = c$ *for* $a, b, c \in E$ *whenever this equation holds in* E*; let* ι *be the natural embedding of* E *in* $\mathcal{U}(E)$*; and let* $u = \iota(1)$*. We may then partially order* $\mathcal{U}(E)$ *by letting* $\mathcal{U}(E)^+$ *the subsemigroup generated by* $\iota(E)$*. Moreover,* $((\mathcal{U}(E), u), \iota)$ *is a universal ambient group for* E*.*

For an interval PE algebra E, we will in the sequel identify E and $\Gamma(\mathcal{U}(E), u)$, which in particular means $u = 1$, and we will refer to $(\mathcal{U}(E), 1)$, or $\mathcal{U}(E)$, simply as the extension of E to its universal ambient group.
Now, under the following condition a PE algebra is always a *po*-group interval (Dvurečenskij and Vetterlein, 2001b, Theorem 5.7).

Definition 2.9. We say that a PE algebra $(E; +, 0, 1)$ fulfils (RDP) if for any $a, b, c, d \in E$ such that $a + b = c + d$, there are $e_1, e_2, e_3, e_4 \in E$ such that the following scheme holds:

$$\begin{array}{cccc} e_1 & e_2 & \rightarrow & a \\ e_3 & e_4 & \rightarrow & b \\ \downarrow & \downarrow & & \\ c & d & & \end{array} \tag{1}$$

Moreover, we say that E fulfils (RDP_1) if for any $a, b, c, d \in E$ such that $a + b = c + d$, there are $e_1, e_2, e_3, e_4 \in E$ such that (i) the scheme (1) holds and (ii) for every $e_2' \leq e_2$ and $e_3' \leq e_3$, we have $e_2' + e_3' = e_3' + e_2'$.

Herein, by the scheme (1) to hold, we mean that every row and every column sums up to what the points to; see Dvurečenskij and Vetterlein (2001a).

Theorem 2.10. *Any PE algebra* $(E; +, 0, 1)$ *fulfilling* (RDP_1) *is an interval PE algebra.*

It is clear how the representing *po*-groups of symmetrically complemented PE algebras are to be characterized.

Recall that the *center* of a group is the subset of those group elements which commute with all other elements.

Lemma 2.11. *Let* $(E; +, 0, 1)$ *be an interval PE algebra and* $(\mathcal{U}(E), 1)$ *the extension of E to its universal ambient group. Then E is symmetrically complemented if and only if* 1 *as an element of* $\mathcal{U}(E)$ *is in the center of* $\mathcal{U}(E)$.

3. STATES ON INTERVAL PE ALGEBRAS

To analyze the structure of some algebra, we may consider the set of all homomorphisms to a certain simple kind of the same type of algebra. The probably simplest nontrivial kind of a PE algebra is given by the real unit interval, that is, by $\Gamma(\mathbb{R}, 1)$, where $\mathbb{R}$ is the additive group of real numbers. The homomorphisms to it are called states (Dvurečenskij and Vetterlein, 2001c).

Definition 3.1. Let $(E; +, 0, 1)$ be a PE algebra. Let $([0, 1]; +, 0, 1)$ be the PE algebra whose ground set is the real unit interval and whose addition is the usual sum of real numbers whenever defined within $[0, 1]$. Then a *state* on E is a homomorphism $s : E \to [0, 1]$ of PE algebras.

Moreover, let (G, u) be a unital *po*-groups. Then a *state* on G is a homomorphism $s : G \to \mathbb{R}$ of the unital *po*-groups (G, u) and $(\mathbb{R}, 1)$.

In other words, a state on a PE algebra E is a function from E to the real unit interval preserving any existing sum and mapping the constant 1 to 1. Similarly, a state on a *po*-group (G, u) is a function from G to the real numbers preserving addition and positivity and mapping u to 1.

So states on a PE algebra E, which is possibly noncommutative, are homomorphisms to the commutative algebra $[0, 1]$. Possible different elements $a + b$ and $b + a$ for some $a, b \in E$, are mapped to the same value. So one may wonder if there is an effect algebra underlying a PE algebra having the same state space. This is, under certain assumptions, indeed the case (Dvusečenskij and Vetterlein, 2001c).

Theorem 3.2. *Let $(E; +, 0, 1)$ be an interval PE algebra possessing a state. Then there is an interval effect algebra $(F; +, 0, 1)$ and an epimorphism $\varepsilon : (E; +, 0, 1) \to (F; +, 0, 1)$ such that the following holds. Any state s on E is of the form $s = s^F \circ \varepsilon$ for some state s^F on F; so the mapping $s \mapsto s^F$ establishes a one-to-one correspondence between the states of E and the states of F.*

Now note that Theorem 3.2 makes one crucial assumption: At least one state on the PE algebra under consideration must exist. Only if this is case, we may reduce a noncommutative PE algebra to a commutative one in a way which preserves the states and, by the way, also the other homomorphisms to commutative PE algebras. Now, to find sufficient conditions under which this situation is given is the motivation underlying this paper.

In this section, we formulate exact conditions for a state on a PE algebra to exist. Let us see first how the states on a PE algebra are related to those on a representing group.

Proposition 3.3. *Let $(E; +, 0, 1)$ be an interval PE algebra, and let $(\mathcal{U}(E), 1)$ be the extension of E to its universal ambient group. Then any state on E may be extended uniquely to a state on $(\mathcal{U}(E), 1)$; and the reduction of any state on $(\mathcal{U}(E), 1)$ to E is a state on E.*

In particular, E possesses a state if and only if $(\mathcal{U}(E), 1)$ *possesses a state.*

Proof: Let $s : E \to [0, 1]$ be a state. Then s, viewed as a mapping from E to $\mathbb{R}$, preserves existing sums, whence by Definition 2.7, s may be extended to a group homomorphism $S : \mathcal{U}(E) \to \mathbb{R}$.
Then S is a state. Indeed, S is, as stated, a group homomorphism; furthermore, it maps $\mathcal{U}(E)^+$ to $\mathbb{R}^+$, because $S(E) = s(E) \subseteq [0, 1]$ and $\mathcal{U}(E)^+$ is generated by E; and, certainly, we have $S(1) = s(1) = 1$. □

For a unital *po*-group, we have in turn the following exact criterion for a state to exist.

Definition 3.4. Let $(G; +, \leq)$ be a *po*-group. Let G_{nc} be the subgroup of G generated by the elements of the form $(a + b) - (b + a)$, $a, b \in G$, endowed with the order inherited from G. We call G_{nc} the *commutator sub-po-group of G*.
Moreover, let (G, u) be a unital *po*-group. Let $G_{\mathrm{nc},u}$ be the subgroup of G generated by u and G_{nc}, endowed with the order inherited from G, We call $(G_{\mathrm{nc},u}, u)$ the *unital commutator sub-po-group of* (G, u).

Remark 3.5. Given a *po*-group G, note how we may describe the elements of G_{nc}. Namely, G_{nc} contains exactly all sums of elements or negated elements of G such that every element occurs the same number of times negated and nonnegated. In particular, G_{nc} is a normal subgroup and thus, endowed with the order of G, a *po*-group.

More general, let H be a subgroup of G containing G_{nc}. If then any sum of elements of G is contained in H, also any permutation of this sum is in H. So again, H is automatically normal and, endowed with the order from G, a *po*-group.
Finally, given a unital *po*-group (G, u) such that u is in the center of G, note how we may describe $G_{\mathrm{nc},u}$. Namely, we simply have $G_{\mathrm{nc},u} = G_{\mathrm{nc}} \cup \{ku : k \in \mathbb{Z}\}$.

Proposition 3.6. *Let* (G, u) *be a unital po-group. Then* (G, u) *possesses a state if and only if there is no* $a \geq u$ *in* G_{nc}.

Proof: Let s be a state on (G, u). Then $a \geq u$ for some $a \in G_{\mathrm{nc}}$ would mean $s(a) \geq 1$; but s is constantly 0 on the whole of G_{nc}. This proves one direction of the statement.

Let now $\mathrm{conv}(G_{\mathrm{nc}})$ be the convex hull of G_{nc} in the partially ordered set G; then $\mathrm{conv}(G_{\mathrm{nc}})$ is closed under the group operations and under conjugation, so it is a convex normal subgroup of G. So we may form the quotient group $[G]_{\mathrm{conv(G_{nc})}}$ of G by $\mathrm{conv}(G_{\mathrm{nc}})$, which has $[u]_{\mathrm{conv(G_{nc})}}$ as a strong unit.

If for no $a \in G_{\mathrm{nc}}$ we have $u \leq a$, then $u \notin \mathrm{conv}(G_{\mathrm{nc}})$, and so $([G]_{\mathrm{conv(G_{nc})}}, [u]_{\mathrm{conv(G_{nc})}})$ is a nontrivial abelian unital *po*-group. Every such group possesses a state by Goodearl (1986, Corollary 4.4). It follows that also (G, u) possesses a state. □

We will now relate the states on a unital *po*-group (G, u) to its unital commutator sub-*po*-group $(G_{\mathrm{nc},u}, u)$. For, we will see that, if there is a state on one of these groups, then there is one on the other one. What we prove is even more: any state on a subgroup of G containing $G_{\mathrm{nc},u}$ is extendible to the whole of G.

We note that this result is in perfect analogy to Goodearl (1986, chap. 4) about abelian *po*-groups fulfilling (RDP) and, furthermore, in accordance with Dvurečenskij (2002b).

Proposition 3.7. *Let (G, u) be a unital po-group such that u is in the center of G.*

(i) Let H be a normal sub-po-group of G containing $G_{\mathrm{nc},u}$. Let s be a state on (H, u). Then s is extendible to a state on (G, u).
(ii) (G, u) possesses a state if and only if $(G_{\mathrm{nc},u}, u)$ possesses a state.

Proof: (i) Let $\mathcal{S}$ be the set of all extensions of the state $s : H \to [0, 1]$ to a state on a larger normal subgroup of G, and let $\mathcal{S}$ be partially ordered by the extension relation. Since then every chain in $\mathcal{S}$ possesses a supremum, there is, by Zorn's Lemma, a maximal element $s' : H' \to [0, 1]$. Let us assume that there is some $c \in G \setminus H'$. We will show that then s' is extendible to a state s'' on the normal subgroup H'' generated by H' and c; this contradicts the maximality of s', and it follows that s' is state on the whole of G.

Since H' contains G_{nc}, it follows from Remark 3.5 that the subgroup generated by H' and c is normal and thus equal to H''. Furthermore, we again conclude from $G_{\mathrm{nc}} \subseteq H'$ that $H'' = \{k + ic : k \in H' \text{ and } i \in \mathbb{Z}\}$.
Let now

$$p = \sup \left\{ \frac{s'(a)}{m} : a \in H', m \in \mathbb{N}, a \leq mc \right\},$$

$$r = \inf \left\{ \frac{s'(b)}{n} : b \in H', n \in \mathbb{N}, nc \leq b \right\}.$$

We claim that then $-\infty < p \leq r < \infty$. Indeed, choose $k \in \mathbb{N}$ such that $-ku \leq c \leq ku$; then $-\infty < -k = s'(-ku) \leq p$, and similarly, we have $r \leq k < \infty$. Furthermore, choose $a, b \in H'$ and $m, n \in \mathbb{N}$ such that $a \leq mc$ and $nc \leq b$. Then

$na \leq mnc \leq mb$, and we conclude

$$\frac{s'(a)}{m} = \frac{s'(na)}{mn} \leq \frac{s'(mb)}{mn} = \frac{s'(b)}{n},$$

which means $p \leq r$.
Now choose $q \in [p, r]$. We claim that we may define

$$s'' : H'' \to \mathbb{R},$$
$$k + ic \mapsto s'(k) + iq, \text{ where } k \in H' \text{ and } i \in \mathbb{Z}.$$

Indeed, assume $k_1 + i_1 c = k_2 + i_2 c$ for some $k_1, k_2 \in H'$, $i_1, i_2 \in \mathbb{Z}$, $i_2 \leq i_1$; this means $-k_1 + k_2 = (i_1 - i_2)c$. In the case $i_1 = i_2$ we have $s'(k_1) = s'(k_2)$, hence $s'(k_1) + i_1 q = s'(k_2) + i_2 q$. Otherwise we have $\frac{s'(-k_1+k_2)}{i_1-i_2} \leq p \leq q \leq r \leq \frac{s'(-k_1+k_2)}{i_1-i_2}$, that is, $s'(-k_1 + k_2) = (i_1 - i_2)q$ or $s'(k_1) + i_1 q = s'(k_2) + i_2 q$.

Moreover, s'' is positive. Indeed, assume $k + ic \geq 0$ for some $k \in H'$ and $i \in \mathbb{Z}$. If then $i < 0$, we have $\frac{s'(k)}{-i} \geq r \geq q$, that is, $s''(k + ic) = s'(k) + iq \geq 0$. If $i = 0$, we have $k \geq 0$, whence $s''(k) = s'(k) \geq 0$. If $i > 0$, we have $\frac{s'(-k)}{i} \leq p \leq q$, that is, $s''(k + ic) = s'(k) + iq \geq 0$. This concludes the proof that s'' is a state on (H'', u).
(ii) This follows easily from part (i). □

Given a group G, let $G^0 = G$ and, for $n \geq 1$, $G^n = G_{\text{nc}}^{n-1}$. Recall that G is called *solvable*, if for some n, $G^n = \{0\}$. An easy consequence of Proposition 3.7 is the following.

Proposition 3.8. *Let (G, u) be a unital po-group such that u is in the center, and let G be solvable. Then G possesses a state.*

Proof: We have $G_{\text{nc},u} = G_{\text{nc}} \cup \{ku : k \in \mathbb{Z}\}$ by Remark 3.5. Now, by repeated application of Proposition 3.7 (ii), we conclude that (G, u) possesses a state iff this is the case for the sub-*po*-group $\{ku : k \in \mathbb{Z}\}$, which evidently is true. □

We shall finally apply our results to PE algebras.

Theorem 3.9. *Let* $(E; +, 0, 1)$ *be a symmetrically complemented interval PE algebra, and let* $(\mathcal{U}(E), 1)$ *be the extension of E to its universal ambient group. Then E the following statements are equivalent:*

(i) *E possesses a state.*
(ii) $(\mathcal{U}(E), 1)$ *possesses a state.*
(iii) *There is no* $a \geq 1$ *in* $\mathcal{U}(E)_{\text{nc}}$.
(iv) $(\mathcal{U}(E)_{\text{nc},1}, 1)$ *possesses a state.*

Proof: (i) $\Leftrightarrow$ (ii). By Proposition 3.3, E has a state iff $(\mathcal{U}(E), 1)$ has a state.

(ii) $\Leftrightarrow$ (iii) This is the content of Proposition 3.6.

(ii) $\Leftrightarrow$ (iv) This holds by Proposition 3.7 (ii), because, by Lemma 2.11, 1 is in the center of $\mathcal{U}(E)$. □

4. LEXICOGRAPHICAL PRODUCT OF AN EFFECT ALGEBRA AND A po-GROUP

We now turn our attention to a kind of PE algebras for which it is natural to study homomorphisms to effect algebras and thus to study states. Independently from that, these special PE algebras are quite interesting in their own right. What we will consider are PE algebras which are built up from a commutative part, namely an effect algebra, and a noncommutative part changing elements, so to say, only in the small. The construction resembles the lexicographical product of a pair of *po*-groups.

In the subsequent chapter, we will discuss certain algebraic conditions on PE algebras, for which the PE algebras introduced here provide the typical examples.

Definition 4.1. Let $(E; +, 0, 1)$ be a PE algebra and $(H; +, \leq)$ a *po*-group. Let

$$E \times_{\text{lex}} H \overset{\text{def}}{=} \{(e, h) \in E \times H : e = 0 \text{ and } h \geq 0 \text{ or } 0 < e < 1 \text{ or } e = 1 \text{ and } h \geq\};$$

and define a partial addition on $E \times_{\text{lex}} H$ componentwise whenever this is possible and leads to a result in $E \times_{\text{lex}} H$. Then $(E \times_{\text{lex}} H; +, (0, 0), (1, 0))$ is called the *lexicographical product* of the PE algebra E and the *po*-group H.

This definition is motivated by the following facts.

Lemma 4.2. *Let* $(E; +, 0, 1)$ *be a PE algebra and* $(H; +, \leq)$ *be a po-group.*

(i) *The lexicographical product* $(E \times_{\text{lex}} H; +, (0, 0), (1, 0))$ *of E and H is a PE algebra.*

(ii) *Let E be an interval PE-algebra. Then so is* $E \times_{\text{lex}} H$. *Indeed, let* $(\mathcal{U}(E), 1)$ *be the extension of E to its universal ambient group, and let* $\mathcal{U}(E) \times_{\text{lex}} H$ *be the usual lexicographical product of po-groups; then* $E \times_{\text{lex}} H$ *is isomorphic to* $\Gamma(\mathcal{U}(E) \times_{\text{lex}} H, (1, 0))$.
Moreover, let H be directed. Then $\mathcal{U}(E) \times_{\text{lex}} H$ *is the extension of* $E \times_{\text{lex}} H$ *to its universal ambient group.*

Proof: (i) It is not difficult to verify the axioms (PE1) to (PE4).

(ii) The first half of what is claimed is obvious.

Assume now that H is directed. We show first that $\mathcal{U}(E \times_{\text{lex}} H)$ and $\mathcal{U}(E) \times H$ are isomorphic groups. From Proposition 2.8 we know that the first group of the

two is generated by all $(e, h) \in E \times_{\text{lex}} H$ and subject to the defining relations $(e_1, h_1) + (e_2, h_2) = (e_3, h_3)$ whenever this equation holds in $E \times_{\text{lex}} H$. Now, by the directedness of H we have $H = H^+ - H^+$; and in $\mathcal{U}(E \times_{\text{lex}} H)$ we have that any $(0, h)$, $h \in H^+$ commutes with any $(e, 0)$, $e \in E$. This means that the group is actually generated by all $(e, 0)$ for $e \in E$ and all $(0, h)$ for $h \in H^+$ under the conditions $(e_1, 0) + (e_2, 0) = (e_3, 0)$ for $e_1, e_2, e_3 \in E$ such that $e_1 + e_2 = e_3$, and $(0, h_1) + (0, h_2) = (0, h_3)$ for $h_1, h_2, h_3 \in H^+$ such that $h_1 + h_2 = h_3$. But then it is clear that $\mathcal{U}(E \times_{\text{lex}} H)$ may be identified with the product of $\mathcal{U}(E)$ and H, that is, with the group $\mathcal{U}(E) \times H$.
Now, under this identification, $\mathcal{U}(E \times_{\text{lex}} H)^+$ is by Proposition 2.8 the subsemigroup generated by the elements (e, h), where $e \in E$, $h \in H$ and either $e > 0$ or else $h \in H^+$. It follows that $\mathcal{U}(E \times_{\text{lex}} H)^+$ is identified with $(\mathcal{U}(E) \times_{\text{lex}} H)^+$. □

We note that for lexicographical products of PE algebras and *po*-groups, symmetric complementation is not unnatural. Besides, we see that a symmetrically complemented PE algebra is not necessarilly commutative.

Proposition 4.3. *The lexicographical product of an effect algebra and a po-group is symmetrically complemented.*

Proof: This is easily checked. □

We shall now establish the exact conditions under which the lexicographical product of a PE algebra and a *po*-group is of the kind we are primarily interested in by fulfilling the Riesz condition (RDP). Note that we will not be concerned with the condition (RDP_1) here. Indeed, (RDP_1) would have rather strong consequences; it would typically imply the abelianess of the *po*-group involved. Anyhow, the more general discussion of the subsequent Section 5 is about interval PE algebras fulfilling (RDP); so it is the latter property which is of primary interest in the present context.

We note that we generalize Corollary 2.12 of Goodearl (1986) to the non-commutative case.

Theorem 4.4. *Let* $(E \times_{\text{lex}} H; +, (0, 0), (1, 0))$ *be the lexicographical product of an at least three-element PE algebra E and a po-group H. Then* $E \times_{\text{lex}} H$ *fulfils* (*RDP*) *if and only if E as well as H fulfil* (*RDP*) *and one of the following conditions holds.*

(α) *E is atomless, and for any pair of noncomparable elements* $a, b \in E$ *whose sum* $a + b$ *exists, there is a nonzero* $x < a, b$ *such that*

$$a/x + b = b/x + a.$$

(β) *For any $g, h, k \in H$ there is an $f \geq g, h$ commuting with k.*
(γ) *E has a smallest nonzero element. Furthermore, for any $g \in H$ there is a positive $f \geq g$ commuting with g.*

Proof: Assume first that $E \times_{\text{lex}} H$ fulfils (RDP). It is clear that then E as well as H fulfil (RDP).

Assume now that (α) does not hold for the reason that there is an atom a. Then either there is a $b \in E$ such that $a \wedge b = 0$, or a is the only atom and lies below all nonzero elements.
In the first case, $(\beta)_1$ follows. Indeed, we have $a + b = b + a$ (Dvurečenskij and Vetterlein (2001a, Lemma 3.2 (ii)). For $g, h, k \in H$, we may apply (RDP) to the equation

$$(a, -g) + (b, k - h) = (b, -g + k) + (a, -h),$$

to get $f^1, \ldots, f^4 \in H$ such that

$$\begin{array}{cccc} (0, f^1) & (a, f^2) & \to & (a, -g) \\ (b, f^3) & (0, f^4) & \to & (b, k-h) \\ \downarrow & \downarrow & & \\ (b, -g+k) & (a, -h). & & \end{array}$$

Because then $f^1, f^4 \geq 0$, we have $f^2 \leq -g, -h$ and so $f \stackrel{\text{def}}{=} -f^2 \geq g, h$; and f^2 commutes with f^3, so f commutes with $f^3 - f^2 = (f^3 + f^4) - (f^2 + f^4) = k$.
In the second case, (γ) follows. Indeed, let $g \in H$. The sum $a + a$ exists, because otherwise E would contain only the two elements 0 and $1 = a$. So we may apply (RDP) to the equation

$$(a, 0) + (a, 0) = (a, g) + (a, -g),$$

to get $f^1, f^2 \in H$ such that either the scheme

$$\begin{array}{cccc} (0, f^1) & (a, f^2) & \to & (a, -g) \\ (a, -f^1) & (0, -f^2) & \to & (a, g) \\ \downarrow & \downarrow & & \\ (a, 0) & (a, 0) & & \end{array}$$

holds, in which case we have $f \stackrel{\text{def}}{=} -f^2 \geq 0, g$ and $f + g = -f^2 - f^1 - f^2 = g + f$; or

$$\begin{array}{cccc} (a, f^1) & (0, f^2) & \to & (a, -g) \\ (0, -f^1) & (a, -f^2) & \to & (a, g) \\ \downarrow & \downarrow & & \\ (a, 0) & (a, 0) & & \end{array}$$

holds, in which case we have $f \stackrel{\text{def}}{=} -f^1 \geq 0$, g and $f + g = g + f$.
Assume now that (α) does not hold for the reason that there is are noncomparable, summable $a, b \in E$ such that for no nonzero $x \leq a, b$, we have $a/x + b = b/x + a$. Choose then $a' \in E$ such that $a + b = b + a'$, and let $e^1, e^2 \in E$ such that

$$\begin{array}{ccccc} e^1 & a \setminus e^1 & \rightarrow & a \\ b \setminus e^1 & e^2 & \rightarrow & b \\ \downarrow & \downarrow & & \\ b & a' & & \end{array}$$

holds. This means $a + b \setminus e^1 + e^2 = b + a \setminus e^1 + e^2$, that is, $a/e^1 + b = b/e^1 + a$; so we have by assumption $e^1 = 0$ and thus $e^2 = 0$ and $a' = a$. We may now proceed as above to prove (β).
This finishes the proof of one direction of the claimed equivalence.
Assume now that E and H fulfil (RDP), and let (α) or (β) or (γ) hold. We are going to prove that (RDP) holds in $E \times_{\text{lex}} H$; so let the equation $(a_1, a_2) + (b_1, b_2) = (c_1, c_2) + (d_1, d_2)$ hold in $E \times_{\text{lex}} H$. Since (RDP) holds in E, we may easily determine the first components of the four elements we are looking for; choose $e_1, \ldots, e_4 \in E$ such that

$$\begin{array}{cccc} e^1 & e^2 & \rightarrow & a_1 \\ e^3 & e^4 & \rightarrow & b_1 \\ \downarrow & \downarrow & & \\ c_1 & d_1. & & \end{array} \tag{2}$$

To determine the appropriate second components from H, we will distinguish several cases, dependent on at which places in (2) zeros appear. The case that all entries in (2) are 0 is easily handled by the fact that (RDP) holds in H.
Assume now that one whole row or column in (2) is 0; a typical example would be $e^1 = e^2 = 0$. If then, furthermore, $e^3 = c^1 > 0$ the scheme

$$\begin{array}{cccc} (0, a_2) & (0, 0) & \rightarrow & (0, a_2) \\ (c_1, -a_2 + c_2) & (d_1, d_2) & \rightarrow & (b_1, b_2) \\ \downarrow & \downarrow & & \\ (c_1, c_2) & (d_1, d_2) & & \end{array}$$

fulfils the requirements. If, on the other hand, $e^3 = c_1 = 0$ and $e^4 = b_1 = d_1 > 0$, we have $a_2, c_2 \geq 0$. We may then apply (RDP) to the equation $a_2 + c_2 = c_2 + (-c_2 + a_2 + c_2)$ to get $f^1, \ldots, f^4 \in H$ such that $f^1 + f^2 = a_2$, $f^3 + f^4 = f^1 + f^3 = c_2$, and $f^2 + f^4 = -c_2 + a_2 + c_2$. Set $f'^4 = -f^3 + b_2 = -f^2 + d_2$, and consider the scheme

$$\begin{array}{cccc} (0,\ f^1) & (0,\ f^2) & \to & (0, a_2) \\ (0,\ f^3) & (b_1,\ f'^4) & \to & (b_1, b_2) \\ \downarrow & \downarrow & & \\ (0, c_2) & (b_1, d_2). & & \end{array}$$

Assume next that in (2), there is a zero in the $e^1 - e^4$ diagonal, for instance $e^1 = 0$, and that $e^2 = a_1 > 0$ and $e^3 = c_1 > 0$.

If (α) holds and a_1, c_1 are either noncomparable or equal, there is some $x \in E$ such that $0 < x < a_1, c_1$ and $a_1/x + c_1 = c_1/x + a_1$. Then the scheme

$$\begin{array}{cccc} (x, a_2) & (a_1 \setminus x, 0) & \to & (a_1, a_2) \\ (c_1 \setminus x, -a_2 + c_2) & (e'^4, d_2) & \to & (b_1, b_2) \\ \downarrow & \downarrow & & \\ (c_1, c_2) & (d_1, d_2) & & \end{array} \tag{3}$$

fulfils, for some nonzero $e'^4 \in E$, the requirements.

If (α) holds and $a_1 < c_1$, set $x = a_1$ and $e'^4 = d_1$ in (3). Similarly, we proceed in the case $c_1 < a_1$.

If (β) holds, then there is an $f^1 \leq a_2, d_2$ such that f^1 commutes with $f^2 \stackrel{\text{def}}{=} f^1 + b_2 - d_2$. We get

$$\begin{array}{cccc} (0, a_2 - f^1) & (a_1,\ f^1) & \to & (a_1, a_2) \\ (c_1,\ f^2) & (e^4, -f^1 + d_2) & \to & (b_1, b_2) \\ \downarrow & \downarrow & & \\ (c_1, c_2) & (d_1, d_2). & & \end{array}$$

If (γ) holds, let a be the atom. Then a commutes with all $b < 1$; indeed, choose a', a'' such that $a + b = b + a'$ and $b + a = a'' + b$; then $a + b = b + a + (a' \setminus a) = (a''/a) + a + b + (a' \setminus a)$, whence $a = a' = a''$. So if $c_1 > a$, set $x = a$ and $e'^4 = x + e^4$ in (3). Similarly, we proceed in the case $c_1 = a$, but $a_1 > a$. If $a_1 = c_1 = a$, choose an $f \in H^+$ such that $f \geq -c_2 + a_2$ and f commutes with $-c_2 + a_2$, and consider the scheme

$$\begin{array}{cccc} (a, a_2 - f) & (0,\ f) & \to & (a, a_2) \\ (0,\ f - a_2, +c_2) & (b_1, -f + d_2) & \to & (b_1, b_2) \\ \downarrow & \downarrow & & \\ (a, c_2) & (b_1, d_2). & & \end{array} \tag{4}$$

Assume next that in (2), there is a zero in the $e^2 - e^3$ diagonal, for instance $e^2 = 0$, and that $e^1 = a_1 > 0$ and $e^4 = d_1 > 0$.

If, moreover, $e^3 > 0$, we have

$$\begin{array}{ccccc} (a_1, a_2) & (0, 0) & \to & (a_1, a_2) \\ (e^3, -a_2 + c_2) & (d_1, d_2) & \to & (b_1, b_2) \\ \downarrow & \downarrow & & \\ (c_1, c_2) & (d_1, d_2). & & \end{array}$$

So let $e^3 = 0$.
If (α) holds, we choose some $x \in E$ such that $0 < x < a_1, d_1$, and we have

$$\begin{array}{ccccc} (a_1, /x, a_2) & (x, 0) & \to & (a_1, a_2) \\ (x, -a_2 + c_2) & (b_1, \backslash x, d_2) & \to & (b_1, b_2) \\ \downarrow & \downarrow & & \\ (a_1, c_2) & (b_1, d_2). & & \end{array}$$

If (β) holds, choose $r, s \in H$ such that $r \geq -a_2, -c_2$ and $s \geq -b_2, -d_2$, and apply (RDP) to the equation $(r + a_2) + (b_2 + s) = (r + c_2) + (d_2 + s)$ to get $f^1, \ldots, f^4 \in H^+$ such that $f^1 + f^2 = r + a_2$, $f^3 + f^4 = b_2 + s$, $f^1 + f^3 = r + c_2$, $f^2 + f^4 = d_2 + s$. Then we have the scheme

$$\begin{array}{ccccc} (a_1, -r + f^1) & (0, f^2) & \to & (a_1, a_2) \\ (0, f^3) & (b_1, f^4 - s) & \to & (b_1, b_2) \\ \downarrow & \downarrow & & \\ (a_1, c_2) & (b_1, d_2) & & \end{array}$$

If (γ) holds, choose an $f \in H^+$ such that $f \geq -c_2 + a_2$ and f and $-c_2 + a_2$ commute, to get a scheme similar to (4).

If in (2) there appears no zero, we have

$$\begin{array}{ccccc} (e^1, a_2) & (e^2, 0) & \to & (a_1, a_2) \\ (e^3, -a_2 + c_2) & (e^4, d_2) & \to & (b_1, b_2) \\ \downarrow & \downarrow & & \\ (c_1, c_2) & (d_1, d_2). & & \end{array}$$

This finishes the proof of the second half of the theorem. □

We have in particular the following.

Theorem 4.5. *Let* $(E \times_{\text{lex}} H; +, (0, 0), (1, 0))$ *be the lexicographical product of a PE algebra E and a po-group H. Then also* $E \times_{\text{lex}} H$ *fulfils* (RDP) *if E as well as H fulfils* (RDP) *and one of the following conditions holds.*

(α) *For any nonzero $a, b \in E$ such that $a + b$ exists, there is an $x \in E$ such that $0 < x < a, b$ and a/x commutes with b/x.*
(β) *Any pair of elements from H has an upper bound in the center of H.*

Now, although the lexicographical products discussed so far involve possibly non-commutative PE algebras, the example we have in mind is the lexicographical product of an effect algebra and a, possibly non-abelian, *po*-group.

It is clear that considering states on such algebras means considering states on the underlying effect algebra. Compare Theorem 3.2.

Proposition 4.6. *Let* $(E \times_{\text{lex}} H; +, (0, 0), (1, 0))$ *be the lexicographical product of an effect algebra E and a directed po-group H. Let* $\iota : E \times_{\text{lex}} H \to E, (a, 0) \mapsto a$*. Then any state s on $E \times_{\text{lex}} H$ is of the form $s = s^E \circ \iota$ for some state s^E on E; so the mapping $s \mapsto s^E$ establishes a one-to-one correspondence between the states of $E \times_{\text{lex}} H$ and the state of E.*

Proof: For any state $s : E \times_{\text{lex}} H \to [0, 1]$, we have $s((0, h)) = 0$ for all $h \in H$, because $n\, s((0, h)) = s((0, nh)) \leq s((1, 0)) = 1$ for all $n \in \mathbb{N}$. Since H is directed, it follows that $s((a, h))$, $h \in H$, does not depend on h. □

5. ORDER REGULAR AND NEARLY COMMUTATIVE PE ALGEBRAS

The lexicographical product of an effect algebra and a possibly non-commutative *po*-group, as studied in the previous section, is just the simplest example of a PE algebra about which we may say that it is noncommutative only in the small. In the present section, we shall make precise this idea, by introducing the appropriate algebraic conditions.

Namely, we shall introduce order regular and nearly commutative PE algebras. These two properties imply, under certain further natural assumptions, that a PE algebra possesses a nontrivial effect algebra as a homomorphic image. In view of Theorem 3.2, this is equivalent to the existence of states on the algebra.

Moreover, we will slightly strengthen the property of being order regular. We will see that this has an amazingly far-reaching consequence; the commutator subgroup of the *po*-group representing the PE algebra is then purely infinitesimal.

Definition 5.1. Let $(E; +, 0, 1)$ be a PE algebra. We shall call $a \in E$ *infinitesimal* if $n\, a$ is defined for all $n \in \mathbb{N}$. We denote the set of all infinitesimal elements of E by E_{infts}.
Furthermore, let (G, u) be a unital *po*-group. We shall call $a \in G$ *infinitesimal* if $n\, a \leq u$ for all $n \in \mathbb{Z}$. We denote the set of all infinitesimal elements of G by G_{infts}.

Now, if the commutator subgroup of the *po*-group representing a PE algebra is purely infinitesimal, the question if the algebra has a state is easily answered affirmatively.

Proposition 5.2. *Let (E; +, 0, 1) be an interval PE algebra, and let $\mathcal{U}(E)$ the extension of E to its universal ambient group. If then the commutator sub-po-group of $\mathcal{U}(E)$ contains only infinitesimal elements, that is, if*

$$\mathcal{U}(E)_{\mathrm{nc}} \subseteq \mathcal{U}(E)_{\mathrm{infts}},$$

then E possesses a state.

Proof: Under the given assumption it is clear that there is no $a \geq u$ in $\mathcal{U}(E)_{\mathrm{nc}}$; so the claim follows from Propositions 3.6 and 3.3. □

For what follows, we need the following preparatory definitions.

Definition 5.3. Let $(E; +, 0, 1)$ be a PE algebra.

(i) Let $a, b \in E$. We say that a is *essentially smaller* than b if $a < b$ and neither the left nor the right difference of b and a is infinitesimal. In this case, we write $a \mathrel{\overline{<}} b$.

(ii) Two elements $a, b \in E$ are called *close* if for any $c \in E$ we have $c \mathrel{\overline{<}} a$ if and only if $c \mathrel{\overline{<}} b$, and if for any $d \in E$ we have $a \mathrel{\overline{<}} d$ if and only if $b \mathrel{\overline{<}} d$. In this case, we write $a \approx b$.

Remark 5.4. It is easily verified that, for elements a, b of a symmetrically complemented interval PE algebra E such that $a \leq b$, we have $a \mathrel{\overline{<}} b$ iff $a \setminus b$ is not infinitesimal iff a / b is not infinitesimal.
On the basis of the notion of closeness, we newly introduce two properties of PE algebras. The algebras of the special type discussed in Section 4 provide typical examples.

Definition 5.5. Let $(E; +, 0, 1)$ be a PE algebra.

(i) E is called *order regular* if any two comparable elements whose left or right difference is infinitesimal, are close.

(ii) E is called *nearly commutative* if, for any $a, b \in E$, $a + b$ exists if and only if $b + a$ exists, in which case we have $a + b \approx b + a$.

Proposition 5.6. *The lexicographical product of an archimedean effect algebra and a po-group is order regular and nearly commutative.*

Let us first state the crucial property of an order regular PE algebra. Recall (Dvurečenskij and Vetterlein, 2001c, Definition 3.4 (i)) that an *ideal* of a PE algebra E is a set $I \subseteq E$ such that (i) for any $b \in I$ and $a \in E$, $a \leq b$ implies $a \in I$ and (ii) for any $a, b \in I$ such that $a + b$ exists, also $a + b \in I$. An ideal is called *normal* if for $a, r, s \in E$ such that $r + a = a + s$, we have $r \in I$ if and only if $s \in I$.

Proposition 5.7. *Let* $(E; +, 0, 1)$ *be an interval PE algebra fulfilling* (RDP) *which is symmetrically complemented and order regular. Then* E_{infts} *is a normal ideal of E. Moreover, any infinitesimal element of E lies below any non-infinitesimal one.*

Proof: Clearly, E_{infts} is closed under smaller elements. To see that E_{infts} is closed under sums, let $a, b \in E_{\mathrm{infts}}$ such that $a + b$ exists. We then have $0 \approx a$, because E is order regular, and $a \not\mathrel{\bar{<}} a + b$; so it follows $0 \not\mathrel{\bar{<}} a + b$, which means that $a + b \in E_{\mathrm{infts}}$. So E_{infts} is an ideal. Because E is symmetrically complemented, it easily follows that E_{infts} is normal.
Let now $a \in E_{\mathrm{infts}}$ and b non-infinitesimal. Then, in view of Remark 5.4, we have $a \mathrel{\bar{<}} a + b$. On the other hand, we have by the order-regularity $b \approx a + b$, so it follows $a \mathrel{\bar{<}} b$ and in particular $a < b$. □

Note that, given a PE algebra E as specified in Proposition 5.7, we may by Dvurečenskij and Vetterlein (2001c, Proposition 3.6) form the quotient algebra $[E]_{E_{\mathrm{infts}}}$, to get a PE algebra which is archimedean.

Here, we are interested in the closeness relation $\approx$, which, as we will see now, proves to be a structure preserving congruence relation as well.

We will from now on assume tacitly that the PE algebras we deal with are nontrivial.

Theorem 5.8. *Let* $(E; +, 0, 1)$ *be an interval PE algebras fulfilling* (RDP) *which is symmetrically complemented and order regular. Then the relation* $\approx$ *on E is a PE algebra congruence; the quotient algebra* $[E]_{\approx}$ *is a nontrivial PE algebra fulfilling (RDP).*

Proof: Let us first note the following. For $a, b \in E$ such that $a \approx b$, we easily see that $a^{\sim} \approx b^{\sim}$. Furthermore, for $a, b \in E$ such that $a + b$ exists, we have $b \approx a^{\sim}$ iff $a + b \approx 1$. Indeed, since $b \leq a^{\sim}$, we have $b \approx a^{\sim}$ iff $(a + b)^{\sim} = a^{\sim} \setminus b \in E_{\mathrm{infts}}$ iff $(a + b)^{\sim} \approx 0$ iff $a + b \approx 1$.
We now prove that $\approx$ is a congruence according to Definition 2.4 (iii). It is, first of all, clear that $\approx$ is an equivalence relation on E. Let now $a, a', b, b' \in E$ be given such that $a + b$ and $a' + b'$ exist and $a \approx a'$ and $b \approx b'$. We claim that $a + b \approx a' + b'$.
In case that $b \approx a^{\sim}$, we have, as shown, $a + b \approx 1$; and from $b' \approx b \approx a^{\sim} \approx a'^{\sim}$, it follows $a' + b' \approx 1 \approx a + b$. In the opposite case, we have $b \mathrel{\bar{<}} a^{\sim}$, which implies

by $a'^{\sim} \approx a^{\sim}$ that $b \mathrel{\bar{<}} a'^{\sim}$ holds and thus $a' + b$ exists. Let now $c \in E$ such that $c \mathrel{\bar{<}} a + b$; we shall see that $c \mathrel{\bar{<}} a' + b$. We then have $c = c_a + c_b$ for some $c_a \leq a$, $c_b \leq b$. If then $a \setminus c_a$ is not infinitesimal, we have $c_a \mathrel{\bar{<}} a$, and it follows $c_a \mathrel{\bar{<}} a'$ and so $c = c_a + c_b \mathrel{\bar{<}} a' + c_b \leq a' + b$. Otherwise, because $c = c_a + c_b \mathrel{\bar{<}} a + b = c_a + a \setminus c_a + b/c_b + c_b$ and so $a \setminus c_a + b/c_b \notin E_{\text{infts}}$ and because E_{infts} is by Proposition 5.7 an ideal, b/c_b is not infinitesimal. Then from $a' \mathrel{\bar{<}} a' + b/c_b$ and $a' \approx a \approx c_a$ we conclude $c_a \mathrel{\bar{<}} a' + b/c_b$ and so $c = c_a + c_b \mathrel{\bar{<}} a' + b$. In a similar manner, we proceed to show that $c \mathrel{\bar{<}} a' + b'$. Finally, we may prove by analogous reasoning that $c \mathrel{\bar{<}} a + b$ iff $c \mathrel{\bar{<}} a' + b'$, and, for any $d \in E$, that $a + b \mathrel{\bar{<}} d$ iff $a' + b' \mathrel{\bar{<}} d$.

We now prove that $\approx$ is actually a PE algebra congruence, using Proposition 2.5. So let $a, a', b \in E$ such that $a + b$ exists and $a' \approx a$; we will show that there is a $b' \approx b$ such that $a' + b'$ exists. We have $a \leq b^{-}$. If even $a \mathrel{\bar{<}} b^{-}$, it follows $a' \mathrel{\bar{<}} b^{-}$ and thus $a' + b$ exists. Otherwise $b^{-} \setminus a$ is infinitesimal; so $a' \approx a \approx b^{-}$ and, setting $b' = a'^{\sim}$, we have $b' = a'^{\sim} \approx b$ and $a' + b' = 1$ exists. Analogously, we see that for any $b'' \approx b$, there is an $a'' \approx a$ such that $a'' + b''$ exists.

So we have proved that $[E]_{\approx}$ is a PE algebra. Now, since the constants 0 and 1 are not close, $[0]_{\approx}$ and $[1]_{\approx}$ are different elements, that is, $[E]_{\approx}$ is nontrivial.

It remains to show that $[E]_{\approx}$ fulfils (RDP); so let $a, b, c, d \in E$ be given such that $[a]_{\approx} + [b]_{\approx} = [c]_{\approx} + [d]_{\approx}$. By what was just proved, there are $b' \approx b$ and $d' \approx d$ such that $a + b'$ and $c + d'$ are defined, and we have $a + b' \approx c + d'$. Now the case that b' is infinitesimal is easy, because then $[a]_{\approx} = [c]_{\approx} + [d]_{\approx}$. Otherwise, we have $b' \mathrel{\bar{>}} 0$, so $a \mathrel{\bar{<}} a + b'$ and consequently $a \mathrel{\bar{<}} c + d'$, that is, $a + b'' = c + d'$ for some $b'' \approx b'$. Thus (RDP), holding in E, enables us to choose the appropriate four elements from $[E]_{\approx}$, as required by Definition 2.9. □

We are finally ready to formulate our first state existing theorem.

Theorem 5.9. *Let* $(E; +, 0, 1)$ *be an interval PE algebra fulfilling (RDP) which is symmetrically complemented, order regular, and nearly commutative. Then the quotient algebra* $[E]_{\approx}$ *is a nontrivial interval effect algebra. In particular, E possesses a state.*

Proof: From Theorem 5.8, we know that $[E]_{\approx}$ is a nontrivial PE algebra fulfilling (RDP).

Now, by Remark 2.3, for any $a, b \in E$, $a + b$ is defined iff $b + a$ is defined, and it follows that $[a]_{\approx} + [b]_{\approx}$ is defined iff $[b]_{\approx} + [a]_{\approx}$ is defined. Because E is nearly commutative, we have $a + b \approx b + a$ if these sums exist. It easily follows that $[E]_{\approx}$ is commutative.

So we have proved that $[E]_\approx$ is an effect algebra fulfilling (RDP). Since for effect algebras, (RDP) and (RDP$_1$) are equivalent, $[E]_\approx$ is by Theorem 2.10 an interval of an abelian *po*-group. Namely, $[E]_\approx$ is the unit interval of $(\mathcal{U}([E]_\approx), [1]_\approx)$.

$(\mathcal{U}([E]_\approx), [1]_\approx)$ possesses by Goodearl (1986, Corollary 4.4) a state, which reduces to a state on $[E]_\approx$. So also E possesses a state. □

Now, on our way to show the existence of states, we did not make use of the strong criterion used in Proposition 5.2, where it was assumed that the commutator sub-*po*-group $\mathcal{U}(E)_{\text{nc}}$ is purely infinitesimal. But interestingly, we may force such a situation, by slightly strengthening one of our conditions.

Definition 5.10. Let $(E; +, 0, 1)$ be a PE algebra. E is called *strongly order regular* if E is order regular and if for any non-infinitesimal close elements $a, b \in E$, there is a non-infinitesimal element $c \leq a, b$.

The typical example for strong order-regularity is given by those lexicographical products of PE algebras and *po*-groups which fulfil the Riesz decomposition property, that is, by those PE algebras which have been characterized in Theorem 4.4.

Proposition 5.11. *The lexicographical product of an archimedean at least five-element PE algebra and a po-group such that this product fulfils* (RDP) *is strongly order regular.*

We need two preparatory lemmas.

Lemma 5.12. *Let* $(E; +, 0, 1)$ *be an interval PE algebra fulfilling* (RDP) *which is symmetrically complemented, order regular and nearly commutative. Let* $\mathcal{U}(E)$ *be the extension of E to its universal ambient group. Then there is a po-group congruence* $\approx$ *on* $\mathcal{U}(E)$ *whose restriction to E is the equally denoted relation on E.*

Proof: Let $\kappa : E \to [E]_\approx$, $a \mapsto [a]_\approx$ the natural homomorphism from E to $[E]_\approx$. By Theorem 5.9, $[E]_\approx$ is an interval effect algebra, represented by $\mathcal{U}([E]_\approx)$. We see from Definition 2.7 and Proposition 2.8 that κ extends to a *po*-group homomorphism from $\mathcal{U}(E)$ to $\mathcal{U}([E]_\approx)$. Now, κ induces a *po*-group congruence $\approx$ on $\mathcal{U}(E)$ which identifies two elements $a, b \in E$ if and only if $\kappa(a) = \kappa(b)$ if and only if $a \approx b$ in E. □

In what follows, we will use the essential order relation $\bar{<}$ for $\mathcal{U}(E)$, the universal ambient group of some PE algebra E, just in the same way as for E: Let, for a pair $a, b \in \mathcal{U}(E)$, $a \bar{<} b$ hold if $a < b$ and $b/a, b \setminus a \notin \mathcal{U}(E)_{\text{infts}}$. If then $a, b \in E$, this relation has obviously the same meaning as with respect to the PE algebra E.

Lemma 5.13. *Let $(E; +, 0, 1)$ be an interval PE algebra fulfilling (RDP) which is symmetrically complemented, strongly order regular, and nearly commutative. Let $\mathcal{U}(E)$ be its extension to its universal ambient group, and let $\approx$ be the extension of the closeness relation from E to $\mathcal{U}(E)$. Then, for $a, b, c \in \mathcal{U}(E)$, $a \mathrel{\bar{<}} b$ and $b \approx c$ imply $a \mathrel{\bar{<}} c$.*

Proof: We shall first prove the following preliminary statement with respect to $\mathcal{U}(E)$: For $b, c \in E$ and $d \in \mathcal{U}(E)^+$, $b \approx c$ and $d \mathrel{\bar{>}} 0$ imply $b \mathrel{\bar{<}} d + c$.

By replacing d by a smaller, but still non-infinitesimal element if necessary, we may assume that $d \in E$. Let $d = d_c + d_{c-}$ such that $d_c \leq c$ and $d_{c-} \leq c^-$. If $d_{c^-} \mathrel{\bar{>}} 0$, then from $c \mathrel{\bar{<}} d_c - +c$ we have $b \mathrel{\bar{<}} d_{c-} + c \leq d + c$. In the other case, we have $d_c \approx d$. If then, in addition $d_c \mathrel{\bar{<}} c$, it follows $d_c \mathrel{\bar{<}} b$, and we may conclude from $c^- \mathrel{\bar{<}} c^- + d_c$ that $c^- \mathrel{\bar{<}} b^- + d_c \leq b^- + d$, whence $b^- + b = c^- + c \mathrel{\bar{<}} b^- + d + c$ and $b \mathrel{\bar{<}} d + c$. If, otherwise, $d_c \approx c \approx d$ and, in addition $c \mathrel{\bar{<}} c^-$, we have $d_c \mathrel{\bar{<}} c^-$ and so $b \approx c \mathrel{\bar{<}} d_c + c$ and $b \mathrel{\bar{<}} d + c$. It remains the case $d_c \approx c \approx c^- \approx d$; then, by strong order-regularity, there is an $e \mathrel{\bar{>}} 0$ such that $e \leq d_c, c^-$, and we have $b \approx c \mathrel{\bar{<}} e + c$ and so $b \mathrel{\bar{<}} d + c$. This completes the proof of the preliminary statement.

Assume now $a \mathrel{\bar{<}} b$ and $b \approx c$, where $a, b, c \in \mathcal{U}(E)$; we shall prove $a \mathrel{\bar{<}} c$. Since $\mathcal{U}(E)$ is directed and the involved relations are translation-invariant, we may assume that $a, b, c \geq 0$. Furthermore, $b \approx c$ then holds exactly if $[b]_\approx = [c]_\approx$ in $\mathcal{U}([E]_\approx)^+$; and $\mathcal{U}([E]_\approx)^+$ is the semigroup freely generated by the elements of E with the defining relations $x + y = z$ for $x, y, z \in E$ such that this equation holds in E, and $x = y$ for $x, y \in E$ such that $x \approx y$ in E. So we may further assume that $b = b_1 + b_2 + b_3$, $c = b_1 + c_2 + b_3$ for $b_2, c_2 \in E$ such that $b_2 \approx c_2$; the general claim follows from this special case by induction.
Let $a = a_1 + a_2 + a_3$ such that $a_1 \leq b_1$, $a_2 \leq b_2$, and $a_3 \leq b_3$. If then $a_2 \mathrel{\bar{<}} b_2$, we have $a_2 \mathrel{\bar{<}} c_2$ and thus $a \mathrel{\bar{<}} c$. Otherwise we have $a_2 \approx b_2$, and it follows $a_1 \mathrel{\bar{<}} b_1$ or $a_3 \mathrel{\bar{<}} b_3$, because E_{infts} is by Proposition 5.7 a normal ideal; suppose e.g. that the first inequality holds. So we have $a_2 \approx c_2$ and $b_1 \setminus a_1 \mathrel{\bar{>}} 0$, and by what was proved above, we know $a_2 \mathrel{\bar{<}} (b_1 \setminus a_1) + c_2$, or $a_1 + a_2 \mathrel{\bar{<}} b_1 + c_2$. It follows $a \mathrel{\bar{<}} c$. □

In view of Proposition 5.2, the subsequent Theorem 5.14 may be considered the strengthened version of a state existence theorem.

Theorem 5.14. *Let $(E; +, 0, 1)$ be an interval PE algebra fulfilling (RDP) which is symmetrically complemented, strongly order regular, and nearly commutative. Let $\mathcal{U}(E)$ be the extension of E to its universal ambient group. Then the commutator sub-po-group of $\mathcal{U}(E)$ contains only infinitesimal elements, that is, we have $\mathcal{U}(E)_{\text{nc}} \subseteq \mathcal{U}(E)_{\text{infts}}$.*

Proof: Let $\approx$, as before, be the extension of the closeness relation from E to $\mathcal{U}(E)$, according to Lemma 5.12. Since $[E]_\approx$ is, by Theorem 5.9, an interval effect algebra, $\mathcal{U}([E]_\approx)$ is abelian, so we have for all $a, b \in \mathcal{U}(E)$ that $[a+b]_\approx = [b+a]_\approx$, or $a+b \approx b+a$. This means for all $g \in \mathcal{U}(E)_{\mathrm{nc}}$ that $g \approx 0$. By $0 \mathrel{\overline{<}} 1$ and Lemma 5.13, it follows $g \mathrel{\overline{<}} 1$, so in particular, $g \leq 1$. We conclude $\mathcal{U}(E)_{\mathrm{nc}} \subseteq \mathcal{U}(E)_{\mathrm{infts}}$. □

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